ALPHA COLLEGE OF ENGINEERING &TECHNOLOGY

FAQ'S OF CALCULUS(2110014)

1 Test the convergence of $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\cdots$

2 Test the convergence of $\sum_{n=1}^{\infty}\frac{n^n}{2^n n!}$

3 Test the convergence of $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+12}\right)^n$

4 Test the convergence of $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\cdots \quad x>0$

5 Discuss the continuity of the following function near origin $f(x,y)=\begin{cases}\frac{x^3y-xy^3}{x^2+y^2} & if\ (x,y)\neq(0,0)\\ 1 & if\ (x,y)=(0,0)\end{cases}$

6 If $u=f(r)$ where $r^2=x^2+y^2$ then prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

7 If $u=f(x-y,y-z,z-x)$ Prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$

8 If $u=\sec^{-1}\left(\frac{x^2+y^2}{x-y}\right)$ then show that $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}=-\cot u\,(2+cot^2u)$

9 If $u=\tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ $show\ that$ $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}=\frac{1}{4}\sin 2u$

10 Find the extreme value of the function $x^3+xy^2+21x-12x^2-2y^2$

11 Find the maxima & minima of $xy-x^2-y^2-2x-2y+4$.

12 Find the point on the plane $x+2y+3z=13$ closest to the point (1, 1, 1).

13 Find the equation of the tangent plane & normal line to the surface $z=2x^2+y^2$ at the point (1, 1, 3).

14 $Evaluate \iint x^2\,dA\ over\ the\ region\ in\ the\ first\ quadrant\ enclosed\ by\ the\ hyperbola\ xy=16$ $y=x, y=0\ and\ x=8.$

15 Change(Reverse) the order of integration and evaluate

$\int_0^1\int_2^{4-2x} dy\,dx$

16 *Find the area of the region R enclosed by the parabola* $y = x^2$ *and the line* $= x + 2$.

17 Check the convergence of $\int_4^{\infty} \frac{3x+5}{x^4+7}\, dx$

18 Evaluate $\lim_{x \to a} \frac{\log(x-a)}{\log(e^x - e^a)}$.

19 If $u = 2xy,\ v = x^2 - y^2,$ where $x = r\cos\theta,\ y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

20 Evaluate $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$

Note: Submit assigned Term work on date 19/11/15.

Q-1 Test the Convergence of $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\cdots$

Sol^n Here $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5} = \sum_{n=1}^{\infty}\frac{2n-1}{n(n+1)(n+2)}$

$\therefore$ Here $a_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n(2-1/n)}{n^3(1+\frac{1}{n})(1+2/n)}$

$a_n = \frac{(2-1/n)}{n^2(1+\frac{1}{n})(1+\frac{2}{n})}$ Choose $b_n = \frac{1}{n^2}$

$\therefore \frac{a_n}{b_n} = \frac{(2-1/n)}{(1+\frac{1}{n})(1+\frac{2}{n})}$ $\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = 2$ (finite & nonzero)

$\therefore \Sigma b_n = \Sigma\frac{1}{n^2}$ is convergent by p-test as $p=2>1$

Hence By limit comparision Test Σa_n is convergent.

Q-2 Test the Convergence of $\sum_{n=1}^{\infty}\frac{n^n}{2^n n!}$

Here $a_n = \frac{n^n}{2^n n!}$, $a_{n+1} = \frac{(n+1)^{n+1}}{2^{n+1}(n+1)!}$

Now $\frac{a_{n+1}}{a_n} = \frac{(n+1)^n(n+1)}{2^{n+1}(n+1)!}\times\frac{2^n n!}{n^n}$

$\frac{a_{n+1}}{a_n} = \frac{\cancel{n^n}(1+1/n)^n\cancel{(n+1)}\,\cancel{2^n}\,\cancel{n!}}{\cancel{2^n}.2\,\cancel{(n+1)}\,\cancel{n!}\,\cancel{n^n}}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{2}(1+\frac{1}{n})^n = \frac{e}{2}>1$

Hence By Ratio Test given series is divergent.

Ex-3 Test the Convergence of $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+12}\right)^n$

Solⁿ Here $a_n = \left(\frac{2n+3}{3n+12}\right)^n$

$$(a_n)^{1/n} = \left[\left(\frac{2n+3}{3n+12}\right)^n\right]^{1/n}$$

$$(a_n)^{1/n} = \frac{2n+3}{3n+12} \quad \therefore \lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\frac{n(2+3/n)}{n\left(3+\frac{12}{n}\right)}$$

$$\therefore \lim_{n\to\infty}(a_n)^{1/n} = \frac{2}{3} < 1$$

$\therefore$ By Cauchy's root test given series is convergent.

Ex-4 Test the Convergence of $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\cdots \quad (x>0)$

Solⁿ $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\cdots = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n^{n-1}}$

let $a_n = \frac{x^{n-1}}{n^{n-1}}$, $a_{n+1} = \frac{x^n}{(n+1)^n}$

$$\frac{a_{n+1}}{a_n} = \frac{x^n}{(n+1)^n} \times \frac{n^n n^{-1}}{x^n x^{-1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{x^n}{n^n\left(1+\frac{1}{n}\right)^n} \times \frac{n^n n^{-1}}{x^n x^{-1}}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{x}{n} = 0 < 1$$

Hence the series converges for all x

By ~~Cauchy's root~~ Ratio Test.

5) Discuss the continuity of the following function near origin

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

$\underline{\text{Sol}^n}$

Path :- 1 : $y = mx$

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3y - xy^3}{x^2+y^2} \right)$$

$$= \lim_{x\to 0} \left(\frac{x^3(mx) - x(mx)^3}{x^2 + m^2x^2} \right)$$

$$= 0$$

Path :- 2 : $y = mx^2$

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3y - xy^3}{x^2+y^2} \right)$$

$$= \lim_{x\to 0} \left(\frac{x^3mx^2 - x(m^3x^6)}{x^2 + m^2x^4} \right)$$

$$= 0$$

Path :- 3 : $y = mx^3$

$$\lim_{(x,y)\to(0,0)} \left[\frac{x^3mx^3 - x(mx^3)^3}{x^2 + (mx^3)^2} \right]$$

$$= 0.$$

Path :- 4 $y = mx^n$

$$\lim_{x\to 0} \left[\frac{x^3mx^n - x(mx^n)^3}{x^2 + (mx^n)^2} \right]$$

$$= 0.$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

$$\& \quad f(0,0) = 1$$

Here

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$$

$\therefore$ $f(x,y)$. is not continuous at $(0,0)$.

(6) If $u = f(r)$ where $r^2 = x^2 + y^2$ then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Sol^n

$$\frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x}$$

$$= f'(r) \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$$

$$\frac{\partial u}{\partial x} = f'(r) \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r) \frac{x^2}{\left(\sqrt{x^2+y^2}\right)^2} + f'(r) \frac{\sqrt{x^2+y^2} - x \frac{1 \cdot 2x}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= f''(r) \frac{x^2}{x^2+y^2} + f'(r) \frac{x^2+y^2-\cancel{x}x^2}{(x^2+y^2)^{3/2}}$$

$$= f''(r) \frac{x^2}{x^2+y^2} + f'(r) \frac{y^2}{(x^2+y^2)(x^2+y^2)^{1/2}}$$

Ily $$\frac{\partial^2 u}{\partial y^2} = f''(r) \frac{y^2}{x^2+y^2} + f'(r) \frac{x^2}{(x^2+y^2)(x^2+y^2)^{1/2}}$$

$$\therefore \text{L.H.S.} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= f''(r)\left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right) + f'(r)\left(\frac{x^2+y^2}{(x^2+y^2)(x^2+y^2)^{1/2}}\right)$$

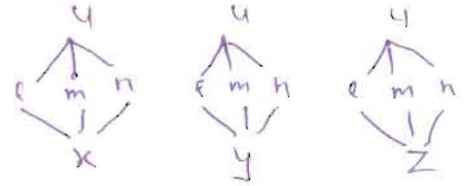
$$= f''(r) + f'(r)\frac{1}{(x^2+y^2)^{1/2}}$$

$$= f''(r) + \frac{1}{r}f'(r)$$

$= R.H.S.$

(7) If $u = f(x-y,\ y-z,\ z-x) = 0$, than Prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

→ Let $x-y = l$, $y-z = m$, $z-x = n$



$u = f(l, m, n)$ —— (1)

Diff. w.r.t 'x'

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot(1) + \frac{\partial u}{\partial m}\cdot(0) + \frac{\partial u}{\partial n}\cdot(-1) \qquad \left[\because \frac{\partial l}{\partial x} = 1, \frac{\partial m}{\partial x} = 0, \frac{\partial n}{\partial x} = -1\right]$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n}$$

Diff. eqn (1) w.r.t 'y'

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}(-1) + \frac{\partial u}{\partial m}(1) + \frac{\partial u}{\partial n}(0)$$

$$= -\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m}$$

Diff. eqn (1) w.r.t 'z'

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot(0) + \frac{\partial u}{\partial m}(-1) + \frac{\partial u}{\partial n}(1)$$

$$= -\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n}$$

$$\text{L.H.S.} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n} - \frac{\partial u}{\partial l} + \frac{\partial u}{\partial m} - \frac{\partial u}{\partial m} + \frac{\partial u}{\partial n}$$

$= 0$

$=$ R.H.S.

(8) If $u = \sec^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$ than show that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = -\cot u(2+\cot^2 u)$

→ $u(x,y) = \sec^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$

$$u(tx, ty) = \sec^{-1}\left(\frac{t^2x^2+t^2y^2}{tx-ty}\right) = \sec^{-1}\left(t^1\left(\frac{x^2+y^2}{x-y}\right)\right)$$

∴ u is not a homogeneous function

Take $\sec u = \frac{x^2+y^2}{x-y} = z$ f

let $f(u) = \sec u$

$$z(tx, ty) = \frac{t^2x^2 + t^2y^2}{tx - ty} = t^1\left(\frac{x^2+y^2}{x-y}\right) = t^1 z(x,y)$$

$\therefore$ z is a homogenous f^n of degree 1.

$\therefore$ By known result,

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = n\,\frac{f(u)}{f'(u)} = 1\cdot\frac{\sec u}{\sec u\cdot\tan u} = \cot u$$

Again by known result,

$$x^2\cdot u_{xx} + 2xy\cdot u_{xy} + y^2 u_{yy} = g(u)\cdot[g'(u)-1]$$

where $g(u) = \frac{n f(u)}{f'(u)}$

$$\therefore\ x^2\cdot u_{xx} + 2xy\cdot u_{xy} + y^2 u_{yy} = \cot u\,[-\operatorname{cosec}^2 u - 1]$$
$$= -\cot u\,[\operatorname{cosec}^2 u + 1]$$
$$= -\cot u\,[1 + \cot^2 u + 1]$$
$$= -\cot u\,[2 + \cot^2 u].$$

(9) If $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show that $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$

$\rightarrow$ $$u(x,y) = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$u(tx, ty) = \tan^{-1}\left(\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right) = \tan^{-1}\left[t^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right]$$

$\therefore$ u is not a homogenous function

Take $\tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = z$ f

let $f(u) = \tan u$

$$z(tx, ty) = \frac{tx + ty}{t^{1/2}x^{1/2} + t^{1/2}y^{1/2}} = t^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) = t^{1/2} z(x,y)$$

$\therefore$ z is a homogenous f^n of degree $1/2$.

$\therefore$ By known result

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)} = \frac{1}{2} \frac{\tan u}{\sec^2 u} = \frac{1}{2} \frac{\sin u}{\cos u} \cdot \cos^2 u$$

$$= \frac{1}{2} \cdot \sin u \cdot \cos u$$

$$= \frac{1}{2 \cdot 2} (2 \sin u \cos u)$$

$$= \frac{1}{4} \sin 2u$$

10) Find the Extreme value of the function
$x^3+xy^2+21x-12x^2-2y^2$

Sol^n $f(x,y) = x^3+xy^2+21x-21x^2-2y^2$

Step: I: for Extreme values $\frac{\partial f}{\partial x} = 0$

i.e $3x^2+y^2-24x+21=0$ —(1)

and $\frac{\partial f}{\partial y} = 0$ $\Rightarrow 2xy-4y=0$
$2y(x-2)=0$
$\Rightarrow y=0,\ x=2$

Putting $y=0$ in (1)
$3x^2-24x+21=0$
$x^2-8x+7=0$
$(x-1)(x-7)=0$
$x=1,\ x=7$ Stationary points are $(1,0),(7,0)$

Putting $x=2$ in (1), $12+y^2-48+21=0$
$y^2=15$ $y=\pm\sqrt{15}$

Stationary points are $(2,\sqrt{15})$. $(2,-\sqrt{15})$

→ Hence all stationary points are $(1,0)$ $(7,0)$ $(2,\sqrt{15})$ and $(2,-\sqrt{15})$

Step II: $r=\frac{\partial^2 u}{\partial x^2} = 6x-24 = 6(x-4)$

$s=\frac{\partial^2 u}{\partial y \partial x} = 2y$ $t=\frac{\partial^2 u}{\partial y^2} = 2x-4 = 2(x-2)$

Step III

(x, y)	r	s	t	$rt - s^2$	Conclusion
$(1, 0)$	-18	0	-2	$36 > 0$ & $r < 0$	maximum
$(7, 0)$	18	0	10	$180 > 0$ and $r < 0$	minimum
$(2, \sqrt{15})$	-12	$2\sqrt{15}$	0	$-60 < 0$	Neither maximum nor minimum
$(2, -\sqrt{15})$	-12	$-2\sqrt{15}$	0	$-60 < 0$	Neither max nor Minimum

Here $f(x,y)$ is maximum at $(1,0)$ and minimum at $(7,0)$

$$f_{max} = 1^3 + (1 \times 0^2) + 21 - (12 \times 1^2) - (2 \times 0^2)$$

$$= \underline{10}$$

$$f_{min} = 7^3 + (7 \times 0^2) + (21 \times 7) - (12 \times 7^2) - (2 \times 0^2)$$

$$= \underline{-98}$$

(11) find maxima and minima of $xy - x^2 - y^2 - 2x - 2y + 4$

Soln $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

step-1 for extreme values $\frac{\partial f}{\partial x} = 0$

$$y - 2x - 2 = 0$$

$$2x - y = -2 \quad \text{— (1)}$$

and $\frac{\partial f}{\partial y} = 0 \Rightarrow x - 2y - 2 = 0$

$$\boxed{x - 2y = 2} \quad \text{— (2)}$$

Multiplying Eqn (1) by (2) and subtracting to (2) eqn

$$4x - 2y = -4$$
$$x - 2y = 2$$
$$- \quad + \quad -$$

$$3x = -6 \qquad \boxed{x = -2}$$

from (1) $2(-2) - y = -2$

$$-4 - y = -2$$

$$y = -4 + 2 = -2 \qquad \boxed{y = -2}$$

$\therefore$ $(-2, -2)$ are stationary points

Step II: $r = \frac{\partial^2 f}{\partial x^2} = -2$

$s = \frac{\partial^2 f}{\partial x \partial y} = 1$

$t = \frac{\partial^2 f}{\partial y} = -2$

$rt - s^2$

$= (-2)(-2) - 1$

$= 4 - 1 = 3 > 0$

$r < 0$, maximum

$f(x, y)$ is maximum at $(-2, -2)$

$$f_{max} = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$$

$$= 4 - 4 - 4 + 4 + 4 + 4$$

$$\underline{\underline{f_{max}}} = \underline{\underline{8}}.$$

(12) Find a point in the plane $x+2y+3z=13$ nearest to the point $(1,1,1)$.

Solⁿ: Let (x,y,z) be a point on the plane $x+2y+3z=13$.
The distance d between (x,y,z) and $(1,1,1)$ is

$$d = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$$

$$d^2 = (x-1)^2+(y-1)^2+(z-1)^2$$

Let $f(x,y,z) = d^2 = (x-1)^2+(y-1)^2+(z-1)^2 \longrightarrow (1)$

$$x+2y+3z=13 \longrightarrow (2)$$

$$\phi(x,y,z) = x+2y+3z-13=0.$$

Let the auxiliary eqⁿ. be

$$(x-1)^2+(y-1)^2+(z-1)^2+\lambda(x+2y+3z-13)=0 \longrightarrow (3)$$

Differentiating eqⁿ. (3) partially w.r.t. x,

$$2(x-1)+\lambda(1)=0$$

$$\therefore \lambda = -2(x-1) \longrightarrow (4)$$

Differentiating Eqⁿ. (3) partially w.r.t. y,

$$2(y-1)+\lambda(2)=0$$

$$\therefore \lambda = -(y-1) \longrightarrow (5)$$

Differentiating Eqⁿ. (3) partially w.r.t. z,

$$2(z-1)+\lambda(3)=0$$

$$\therefore \lambda = -\frac{2}{3}(z-1) \longrightarrow (6)$$

From eqⁿ. (3), (4) & (5),

$$-2(x-1) = -(y-1) = -\frac{2}{3}(z-1)$$

Now, $-2(x-1) = -(y-1)$

$$\Rightarrow 2x-2 = y-1$$

$$\Rightarrow \boxed{y = 2x-1}$$

Also, $-2(x-1) = -\frac{2}{3}(z-1)$,

$$\Rightarrow x-1 = \frac{1}{3}(z-1)$$

$$\Rightarrow 3x-3 = z-1$$

$$\Rightarrow \boxed{z = 3x-2}$$

Substituting y, z in eq^n (2).

$x + 2y + 3z = 13$

$\Rightarrow x + 2(2x-1) + 3(3x-2) = 13$

$\Rightarrow x + 4x - 2 + 9x - 6 - 13 = 0$

$\Rightarrow 14x - 21 = 0$

$\Rightarrow \boxed{x = \frac{3}{2}}$

Ily, $y = 2x - 1$

$\therefore y = 2\left(\frac{3}{2}\right) - 1$

$\therefore \boxed{y = 2}$

And, $z = 3x - 2$

$\therefore z = 3\left(\frac{3}{2}\right) - 2$

$\therefore \boxed{z = \frac{5}{2}}$

Thus, $(x, y, z) = \left(\frac{3}{2}, 2, \frac{5}{2}\right)$ is a point in the plane $x + 2y + 3z = 13$ nearest to the point $(1, 1, 1)$.

(13.) Find the eq^n of the tangent plane and normal line to the Surface $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Sol^n. Let $f(x, y, z) = z - 2x^2 - y^2$

$f_x(x, y, z) = -4x \Rightarrow f_x(1, 1, 3) = -4$

$f_y(x, y, z) = -2y \Rightarrow f_y(1, 1, 3) = -2$

$f_z(x, y, z) = 1 \Rightarrow f_z(1, 1, 3) = 1$

Hence, the eq^n to the tangent plane at point $(1, 1, 3)$ is

$-4(x-1) - 2(y-1) + 1(z-3) = 0$

ie. $-4x + 4 - 2y + 2 + z - 3 = 0$

ie. $-4x - 2y + z + 3 = 0$

ie. $\boxed{4x + 2y - z = 3}$

The eq^n of the normal line is

$$\frac{x-1}{-4} = \frac{y-1}{-2} = \frac{z-3}{+1}$$

ie. $$\boxed{\frac{x-1}{4} = \frac{y-1}{2} = \frac{z-3}{-1}}$$

(14) Evaluate $\iint x^2\, dA$ over the region in the first quadrant enclosed by the hyperbala $xy = 16$, $y = x$, $y = 0$ and $x = 8$

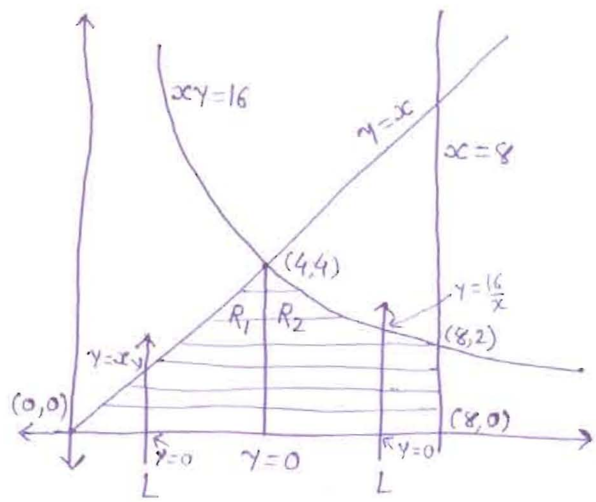
Sol^n:- $xy = 16$

$x = 1,\ y = 16 \quad (1, 16)$

$y = 1,\ x = 16 \quad (16, 1)$

$x = 8,\ y = 2 \quad (8, 2)$

$y = 8,\ x = 2 \quad (2, 8)$



First find intersection paints

$xy = 16$ & $y = x$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

Thus $y = \pm 4$

Consider paint $(4, 4)$ b'coz first quadrant.

For R_1

$$I_1 = \int_0^4 \int_0^x x^2\, dy\, dx = \int_0^4 x^2 [y]_0^x\, dx = \int_0^4 x^2 \cdot x\, dx = \int_0^4 x^3\, dx$$

$$= \left[\frac{x^4}{4}\right]_0^4 = \frac{4^4}{4} = 64$$

For R_2

$$I_2 = \int_4^8 \int_0^{16/x} x^2\, dy\, dx = \int_4^8 x^2 [y]_0^{16/x}\, dx = \int_4^8 x^2 \frac{16}{x}\, dx$$

$$= \int_4^8 16x\, dx = 16\left[\frac{x^2}{2}\right]_4^8 = 8[x^2]_4^8 = 8(64 - 16) = 8 \times 48$$
$$= 384$$

Hence, $I = I_1 + I_2 = 64 + 384 = 448$

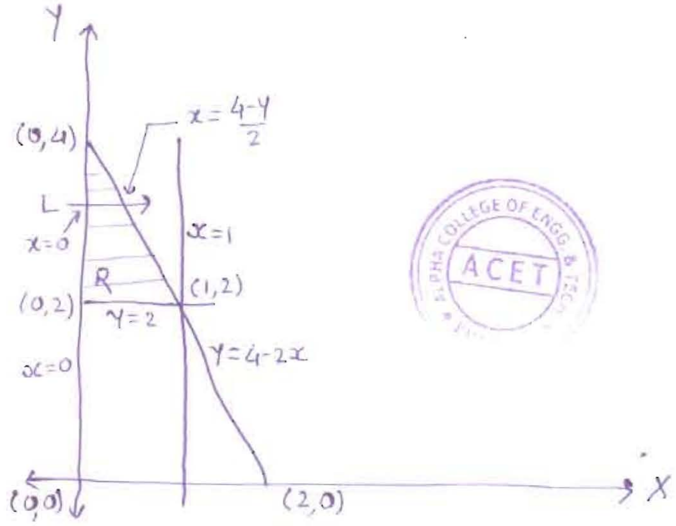
(15) change the order of integration and evaluate

$$\int_0^1 \int_2^{4-2x} dy\, dx$$

→ The region of integration is bounded by.

$y = 2, \quad y = 4-2x$

$x = 0, \quad x = 1$



$y = 4-2x$

(0, 4)

(1, 2)

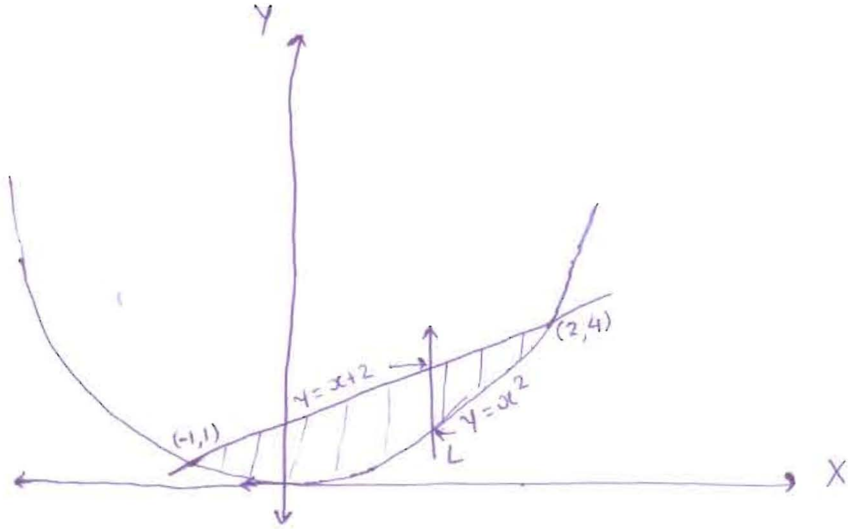
(2, 0)

Hence, the given integral after change of order is

$$\int_0^1 \int_2^{4-2x} dy\, dx = \int_2^4 \int_0^{\frac{4-y}{2}} dx\, dy = \int_2^4 [x]_0^{\frac{4-y}{2}} dy = \int_2^4 \frac{4-y}{2}\, dy$$

$$= \frac{1}{2}\left[4y - \frac{y^2}{2}\right]_2^4 = \frac{1}{2}\left[(16-8)-(8-2)\right] = \frac{1}{2}(2) = 1$$

(16) Find the area of the region R enclosed by the parabala $y = x^2$ and the line $y = x+2$



Now, the intersection of $y = x^2$ & $y = x+2$ is

$x^2 = x+2$

$\therefore x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2,\ x = -1$

Thus, $x = 2 \Rightarrow y = x+2 \Rightarrow y = 4$, i.e. $(2,4)$

$x = -1 \Rightarrow y = x+2 \Rightarrow y = 1$, i.e. $(-1,1)$

Hence $A = \int_{-1}^{2}\int_{x^2}^{x+2} 1\, dy\, dx = \int_{-1}^{2} [y]_{x^2}^{x+2}\, dx = \int_{-1}^{2} (x+2-x^2)\, dx$

$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \left[\left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$

$= \left[2+4-\frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right] = \frac{15}{2} - 3 = \frac{9}{2}$

(17) check the convergence of $\int_{4}^{\infty} \frac{3x+5}{x^4+7}\, dx$

$\rightarrow$ Let $f(x) = \frac{3x+5}{x^4+7}$ & Take $g(x) = \frac{1}{x^3}$

Both $f(x)$ & $g(x)$ are continuous on $[4,\infty)$.

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{x(3+\frac{5}{x})}{x^4(1+7/x^4)}}{1/x^3} = \lim_{x\to\infty} \frac{3+\frac{5}{x}}{1+\frac{7}{x^4}} = 3$

$\therefore \int_{4}^{\infty} f(x)\, dx$ & $\int_{4}^{\infty} g(x)\, dx$ both converges or diverges together.

But $\int_{4}^{\infty} \frac{1}{x^3}\, dx$ is convergent if $P = 3 > 1$

$\therefore \int_{4}^{\infty} f(x)\, dx$ is convergent

(18) Evaluate $\lim_{x \to a} \frac{\log(x-a)}{\log(e^x - e^a)}$

$\rightarrow$ Let $l = \lim_{x \to a} \frac{\log(x-a)}{\log(e^x - e^a)} \quad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x \to a} \frac{\frac{1}{x-a}}{\frac{1}{e^x - e^a} \cdot e^x} \quad \text{(Applying L'Hospital)}$$

$$= \lim_{x \to a} \frac{e^x - e^a}{x-a} \cdot \frac{1}{e^x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{e^x}{1} \quad \lim_{x \to a} \frac{1}{e^x} \quad \text{(L'Hospital)}$$

$$= \frac{e^a}{e^a}$$

$$= 1$$

(19) If $u = 2xy$, $v = x^2 - y^2$, where $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

$\rightarrow$ If $u = 2xy$ & $v = x^2 - y^2$

$u_x = 2y$ $\quad$ $v_x = 2x$

$u_y = 2x$ $\quad$ $v_y = -2y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4y^2 - 4x^2$$

$$= -4(y^2 + x^2)$$

$$= -4r^2$$

Also, $x = r\cos\theta$ & $y = r\sin\theta$

$x_r = \cos\theta$ $\quad$ $y_r = \sin\theta$

$x_\theta = -r\sin\theta$ $\quad$ $y_\theta = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Hence $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)} = -4r^2 \cdot r = -4r^3$

(20) Evaluate $\displaystyle\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$

Sol^n: $\displaystyle\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx = \int_0^1 \int_0^{2-x} [z]_0^{2-x-y}\, dy\, dx$

$$= \int_0^1 \int_0^{2-x} (2-x-y)\, dy\, dx$$

$$= \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^1 \left[2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right] dx$$

$$= \int_0^1 \left[4 - 2x - 2x + x^2 - \frac{(2-x)^2}{2}\right] dx$$

$$= \left[4x - x^2 - x^2 + \frac{x^3}{3} + \frac{(2-x)^3}{6}\right]_0^1$$

$$= \left(4 - 2 + \frac{1}{3} + \frac{1}{6}\right) - \left(\frac{4}{3}\right)$$

$$= \frac{7}{3} + \frac{1}{6} - \frac{8}{6} = \frac{7}{6}$$